

# Indices of collections of equivariant 1-forms and characteristic numbers

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## Abstract

If two  $G$ -manifolds are  $G$ -cobordant then characteristic numbers corresponding to the fixed point sets (submanifolds) of subgroups of  $G$  and to normal bundles to these sets coincide. We construct two analogues of these characteristic numbers for singular complex  $G$ -varieties where  $G$  is a finite group. They are defined as sums of certain indices of collections of 1-forms (with values in the spaces of the irreducible representations of subgroups). These indices are generalizations of the GSV-index (for isolated complete intersection singularities) and the Euler obstruction respectively.

## Introduction

Two closed manifolds are (co)bordant if and only if all their Stiefel–Whitney characteristic numbers coincide. Unitary ( $U$ -) manifolds, that is manifolds with complex structures on the stable normal bundles, are generalizations of complex analytic manifolds adapted to (co)bordism theory. Two closed  $U$ -manifolds are (co)bordant if and only if all their Chern characteristic numbers coincide. (For the facts from cobordism theory see, e.g., [10].) Analogues of Chern numbers for (compact) singular complex analytic varieties were considered in [3] and [4]. In [3] they were defined for compact complex analytic varieties with isolated complete intersection singularities (ICIS). In [4] (other) characteristic numbers were defined for arbitrary compact analytic varieties. (For non-singular varieties both sets of characteristic numbers coincide with the usual Chern numbers.) A survey of these and adjacent results can be found in [5].

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Each characteristic number from any of the sets is the sum of indices of singular (or special) points of (generic) collections of 1-forms defined in [3] and [4]. The indices from [3] are generalizations of the GSV-index of a 1-form defined initially for vector fields in [6] and [8]. The indices from [4] are generalizations of the Euler obstruction defined in [7]. Chern characteristic numbers of (closed) complex analytic manifolds cannot be arbitrary: they satisfy some divisibility conditions. In [2] it was shown that the characteristic numbers of singular analytic varieties from [4] can have arbitrary sets of values.

Let us consider varieties (manifolds) with actions of a finite group  $G$ . Let  $V$  be a  $U$ -manifold (for example, a complex analytic manifold of (complex) dimension  $d$ , that is with real dimension  $2d$ ) with a  $G$ -action respecting the  $U$ -structure. This means that the trivial extension of the  $G$ -action on the tangent bundle  $TV$  to its stabilization preserves the complex structure on it. For a subgroup  $H \subset G$ , let  $V^H$  be the set of fixed points of the subgroup  $H$ . The manifold  $V^H$  is a  $U$ -manifold, i.e. its stable tangent bundle has a natural complex structure. The normal bundle of  $V^H$  has the structure of a complex vector bundle and the group  $H$  acts on it. Let  $r(H)$  be the set of the isomorphism classes of the complex irreducible representations of the group  $H$  and let  $r_+(H) \subset r(H)$  be the set of non-trivial ones. For a component  $N$  of  $V^H$ , the normal bundle of  $V$  is the direct sum of vector bundles of the form  $\alpha \otimes \nu_\alpha$  where  $\alpha$  runs over all the irreducible non-trivial representations of  $H$  and  $\nu_\alpha$  are complex vector bundles (with trivial  $H$ -actions). Moreover 
$$\sum_{\alpha \in r_+(H)} \text{rk } \alpha \cdot \text{rk } \nu_\alpha = (\dim_{\mathbb{R}} V - \dim_{\mathbb{R}} V^H)/2.$$
 For a subgroup  $H \subset G$  and for a

collection  $\underline{n} = \{n_\alpha | \alpha \in r_+(H)\}$ , let  $V^{\underline{n}}$  be the union of the components of  $V^H$  for which the rank of the corresponding vector bundle  $\nu_\alpha$  is equal to  $n_\alpha$ .

Let  $V$  be compact of real dimension  $2d$ , let  $\underline{n} = \{n_\alpha | \alpha \in r_+(H)\}$  be a collection of integers, let  $n_1 := d - \sum_{\alpha \in r_+(H)} n_\alpha \text{rk } \alpha$ , and let  $\nu_1$  be the stable

tangent bundle to  $V^{\underline{n}}$ . In the sequel, the collection  $\underline{n}$  will contain  $n_1$  as well. For a collection of integers  $\underline{k} = \{k_{\alpha,i} | \alpha \in r(H), i=1, \dots, s_\alpha\}$  with  $\sum_{\alpha,i} k_{\alpha,i} =$

$n_1$ , let  $c_{H,\underline{n},\underline{k}}(V, G) = \langle \prod_{\alpha,i} c_{k_{\alpha,i}}(\nu_\alpha^*), [V^{\underline{n}}] \rangle$  be the corresponding characteristic number ( $[V^{\underline{n}}] \in H_{2n_1}(V^{\underline{n}})$  is the fundamental class of the manifold  $V^{\underline{n}}$ ). One can assume that  $k_{\alpha,i} \leq n_\alpha$  for all  $(\alpha, i)$ . These characteristic numbers are invariants of the bordism classes of  $U$ -manifolds with  $G$ -actions, i.e. if two  $G$ - $U$ -manifolds  $(V_1, G)$  and  $(V_2, G)$  are (co)bordant, then, for any  $H \subset G$ ,  $\underline{n}$  and  $\underline{k}$ , the corresponding characteristic numbers  $c_{H,\underline{n},\underline{k}}(V_1, G)$  and  $c_{H,\underline{n},\underline{k}}(V_2, G)$  coincide. One can see that these numbers depend only on the conjugacy class of the subgroup  $H$  (modulo the corresponding identification of the sets of representations of conjugate subgroups).

In this paper we construct, for compact complex singular  $G$ -varieties, two analogues of these characteristic numbers defined as sums of certain indices of collections of 1-forms (with values in the spaces of the irreducible representations of  $H$ ). These indices are generalizations of the GSV-indices and the Chern obstructions defined in [3] and in [4] respectively.

## 1 Characteristic numbers in terms of indices of collections of 1-forms

Let us give a convenient description of the characteristic numbers defined above for complex analytic manifolds. Let  $V$  be a compact complex analytic manifold of (complex) dimension  $d$ . We keep the notations of the Introduction. For each pair  $(\alpha, i)$ , let  $\omega_j^{\alpha, i}$ ,  $j = 1, \dots, n_\alpha - k_{\alpha, i} + 1$ , be generic (continuous) sections of the bundle  $\nu_\alpha^*$  (i.e. linear forms on  $\nu_\alpha$ ). The characteristic number  $c_{H, \underline{n}, \underline{k}}(V, G)$  can be computed as the algebraic (i.e. counted with signs) number of points of  $V^{\underline{n}}$  where for each pair  $(\alpha, i)$  the 1-forms  $\omega_1^{\alpha, i}, \dots, \omega_{n_\alpha - k_{\alpha, i} + 1}^{\alpha, i}$  are linearly dependent.

For arbitrary sections  $\omega_j^{\alpha, i}$ ,  $j = 1, \dots, n_\alpha - k_{\alpha, i} + 1$ , a point  $x \in V^{\underline{n}}$  will be called *singular* if for each pair  $(\alpha, i)$  the 1-forms  $\omega_1^{\alpha, i}, \dots, \omega_{n_\alpha - k_{\alpha, i} + 1}^{\alpha, i}$  are linearly dependent at this point. Assume that all singular points of a collection  $\{\omega_j^{\alpha, i}\}$  are isolated. In this case the characteristic number  $c_{H, \underline{n}, \underline{k}}(V, G)$  is the sum of certain indices of the collection  $\{\omega_j^{\alpha, i}\}$  at the singular points. Let us describe these indices as degrees of certain maps, as intersection numbers, and, in the case when all the 1-forms  $\omega_j^{\alpha, i}$  are complex analytic, as dimensions of certain algebras.

For some simplicity, assume first that the group  $G$  is abelian and therefore all the representations  $\alpha \in r(H)$  are one-dimensional (i.e. they are elements of the group  $H^* = \text{Hom}(H, \mathbb{C}^*)$  of characters of  $H$ ). The linear functions  $\omega_j^{\alpha, i}$  on  $\nu_\alpha$  can be regarded as linear functions on  $T_x V$  for  $x \in V^{\underline{n}}$  assuming that  $\omega_j^{\alpha, i}|_{\beta \otimes \nu_\beta} = 0$  for  $\beta \neq \alpha$ . The form  $\omega_j^{\alpha, i}$  on  $T_x V$  satisfies the condition

$$\omega_j^{\alpha, i}(gu) = \alpha(g)\omega_j^{\alpha, i}(u) \quad (1)$$

for  $g \in H$ ,  $u \in T_x V$ . The forms  $\omega_j^{\alpha, i}$  can be extended to 1-forms on  $V$  satisfying (1). If the group  $G$  is not assumed to be abelian and thus among the irreducible representations  $\alpha$  of the subgroup  $H$  one may have those of dimension higher than 1, the form  $\omega_j^{\alpha, i}$  should be considered as a 1-form on  $V$  with values in the space  $E_\alpha$  of the representation  $\alpha$  satisfying the same condition (1) (where now  $\alpha(g)$  is not a number, but the corresponding operator on  $E_\alpha$ ). The latter simply means that the form  $\omega_j^{\alpha, i}$  is  $H$ -equivariant.

For natural numbers  $p$  and  $q$  with  $p \geq q$ , let  $M_{p,q}$  be the space of  $p \times q$  matrices with complex entries, and let  $D_{p,q}$  be the subspace of  $M_{p,q}$  consisting of matrices of rank less than  $q$ . The complement  $W_{p,q} = M_{p,q} \setminus D_{p,q}$  is the Stiefel manifold of  $q$ -frames (collections of  $q$  linearly independent vectors) in  $\mathbb{C}^p$ . The subset  $D_{p,q}$  is an irreducible complex analytic subvariety of  $M_{p,q}$  of codimension  $p - q + 1$ . Therefore  $W_{p,q}$  is  $2(p - q)$ -connected,  $H_{2p-2q+1}(W_{p,q}; \mathbb{Z}) \cong \mathbb{Z}$ , and one has a natural choice of a generator of this homology group: the boundary of a small ball in a smooth complex analytic slice transversal to  $D_{p,q}$  at a non-singular point.

For collections  $\underline{n} = \{n_\alpha\}$  and  $\underline{k} = \{k_{\alpha,i}\}$  with  $\sum_{\alpha,i} k_{\alpha,i} = n_1$ ,  $k_{\alpha,i} \leq n_\alpha$  for all  $(\alpha, i)$ , let  $M_{\underline{n}, \underline{k}} := \prod_{\alpha,i} M_{n_\alpha, n_\alpha - k_{\alpha,i} + 1}$ ,  $D_{\underline{n}, \underline{k}} := \prod_{\alpha,i} D_{n_\alpha, n_\alpha - k_{\alpha,i} + 1} \subset M_{\underline{n}, \underline{k}}$ . The subset  $D_{\underline{n}, \underline{k}}$  is an irreducible subvariety of (the affine space)  $M_{\underline{n}, \underline{k}}$  of codimension  $\sum_{\alpha,i} k_{\alpha,i} (= n_1)$ , the complement  $W_{\underline{n}, \underline{k}} = M_{\underline{n}, \underline{k}} \setminus D_{\underline{n}, \underline{k}}$  is  $(2n_1 - 2)$ -connected,  $H_{2n_1-1}(W_{\underline{n}, \underline{k}}; \mathbb{Z}) \cong \mathbb{Z}$ , and one has a natural choice of a generator of this homology group. This choice defines the degree (an integer) of a map from an oriented manifold of dimension  $2n_1 - 1$  to  $W_{\underline{n}, \underline{k}}$ .

Assume that the subgroup  $H$  acts on

$$\mathbb{C}^d = \bigoplus_{\alpha \in r(H)} n_\alpha E_\alpha = \bigoplus_{\alpha \in r(H)} \bigoplus_{s=1}^{n_\alpha} E_\alpha^{(s)}$$

by the representation  $\sum_{\alpha \in r(H)} n_\alpha \alpha$  ( $\sum n_\alpha \text{rk } \alpha = d$ ). Here  $E_\alpha^{(s)}$ ,  $s = 1, \dots, n_\alpha$  are copies of the space  $E_\alpha$  of the representation  $\alpha$ . (This means that as a germ of an  $H$ -set  $(\mathbb{C}^d, 0)$  is isomorphic to  $(V, x)$  for  $x \in V^n$ .) The corresponding components of a vector from  $\mathbb{C}^d$  will be denoted by  $u_s^\alpha$  with  $1 \leq s \leq n_\alpha$ ,  $u_s^\alpha \in E_\alpha$ . The vector with the components  $u_s^\alpha$  will be denoted by  $[u_s^\alpha]$ . For a collection  $\underline{k} = \{k_{\alpha,i}\}$  with  $\sum_{\alpha,i} k_{\alpha,i} = n_1$ ,  $k_{\alpha,i} \leq \ell_\alpha$  for all  $(\alpha, i)$ , let  $\omega_j^{\alpha,i}$ ,  $j = 1, \dots, n_\alpha - k_{\alpha,i} + 1$ , be  $H$ -equivariant (i.e. satisfying the condition (1)) 1-forms on a neighbourhood of the origin in  $\mathbb{C}^d$  with values in the spaces  $E_\alpha$ . According to Schur's lemma, at a point  $p \in n_1 E_1 \subset \mathbb{C}^d$  ( $\mathbf{1}$  is the trivial representation of  $H$  and  $E_1$  is its space: the complex line), the 1-form  $\omega_{j|p}^{\alpha,i}$  vanishes on  $\bigoplus_{\beta \neq \alpha} n_\beta E_\beta$  and on each copy  $E_\alpha^{(s)}$  ( $s = 1, \dots, n_\alpha$ ) it is the multiplication by a (complex) number (depending on  $p$ ). Thus let  $\omega_{j|p}^{\alpha,i}([u_s^\beta]) = \sum_s \psi_{j,s}^{\alpha,i}(p) u_s^\alpha$ . Let  $\Psi$  be the map from  $E_1^{n_1} = (\mathbb{C}^n)^H$  to  $M_{\underline{n}, \underline{k}}$  which sends a point  $p \in E_1^{n_1}$  to the collection

of  $n_\alpha \times (n_\alpha - k_{\alpha,i} + 1)$  matrices

$$\left\{ \begin{pmatrix} \psi_{1,1}^{\alpha,i}(p) & \cdots & \psi_{n_\alpha - k_{\alpha,i} + 1, 1}^{\alpha,i}(p) \\ \vdots & \cdots & \vdots \\ \psi_{1, n_\alpha}^{\alpha,i}(p) & \cdots & \psi_{n_\alpha - k_{\alpha,i} + 1, n_\alpha}^{\alpha,i}(p) \end{pmatrix} \right\},$$

whose columns consist of the components  $\psi_{j,s}^{\alpha,i}(p)$  of the 1-forms  $\omega_{j|p}^{\alpha,i}$ . Assume that the collection of the forms has no singular points on  $V$  outside of the origin (in a neighbourhood of it). This means that  $\Psi^{-1}(D_{\underline{n}, \underline{k}}) = \{0\}$ .

If  $\Psi^{-1}(D_{\underline{k}}) = \{0\}$ , the origin in  $E_1^{n_1} \subset \mathbb{C}^d$  is an isolated singular point of the collection  $\{\omega_j^{\alpha,i}\}$ . In this case let us define the *index*  $\text{ind}_{\mathbb{C}^d, 0}^{H, \underline{n}, \underline{k}} \{\omega_j^{\alpha,i}\}$  of the singular point 0 of the collection  $\{\omega_j^{\alpha,i}\}$  as the degree of the map  $\Psi|_{S^{2n_1-1}} : S^{2n_1-1} \rightarrow W_{\underline{n}, \underline{k}}$  or, what is the same, as the intersection number of the image  $\Psi(E_1^{n_1})$  with  $D_{\underline{n}, \underline{k}}$  at the origin.

The origin is a *non-degenerate* singular point of the collection  $\{\omega_j^{\alpha,i}\}$  if the map  $\Psi$  is transversal to the variety  $D_{\underline{n}, \underline{k}}$  at a non-singular point of it. The index of a non-degenerate singular point is equal to  $\pm 1$ . If all the forms  $\omega_j^{\alpha,i}$  are complex analytic, the index is equal to  $+1$ .

The following statement is a reformulation of the definition of the Chern classes of a vector bundle as obstructions to the existence of several linearly independent sections of the bundle. Let  $V$  be a complex analytic manifold of dimension  $d$  with a  $G$ -action, let  $H$  be a subgroup of  $G$ , let  $V^H$  be the set of fixed points of the subgroup  $H$  and, for a collection  $\underline{n} = \{n_\alpha\}$ , let  $V^{\underline{n}}$  be the union of the corresponding components of  $V^H$ . For  $\underline{k} = \{k_{\alpha,i}\}$ , let  $\{\omega_j^{\alpha,i}\}$ ,  $j = 1, \dots, n_\alpha - k_{\alpha,i} + 1$ , be a collection of equivariant 1-forms on  $V$  with values in the spaces  $E_\alpha$  with only isolated singular points on  $V^{\underline{n}}$ . Then the sum of the indices  $\text{ind}_{V, P}^{H, \underline{n}, \underline{k}} \{\omega_j^{\alpha,i}\}$  of these points is equal to the characteristic number  $c_{H, \underline{n}, \underline{k}}(V, G)$ .

Assume that in the situation described above all the 1-forms  $\omega_j^{\alpha,i}$  on  $(\mathbb{C}^d, 0)$  are complex analytic. One has the following statement.

**Proposition 1** *The index  $\text{ind}_{\mathbb{C}^d, 0}^{H, \ell, \underline{k}} \{\omega_j^{\alpha,i}\}$  is equal to the dimension of the factor-algebra of the algebra  $\mathcal{O}_{\mathbb{C}^{n_1}, 0}$  of germs of holomorphic functions on  $(\mathbb{C}^{n_1}, 0)$  by the ideal generated by the maximal (i.e. of size  $(n_\alpha - k_{\alpha,i} + 1) \times (n_\alpha - k_{\alpha,i} + 1)$ ) minors of the matrices  $(\psi_{j,s}^{\alpha,i}(p))$ .*

The proof repeats the one in [3].

## 2 Equivariant GSV-indices and characteristic numbers

Let

$$\mathbb{C}^N = \bigoplus_{\alpha \in r(H)} m_\alpha E_\alpha = \bigoplus_{\alpha \in r(H)} \bigoplus_{s=1}^{m_\alpha} E_\alpha^{(s)}$$

be the complex vector space with the representation  $\sum_{\alpha \in r(H)} m_\alpha \alpha$  of a finite group  $H$  ( $N = \sum m_\alpha \dim E_\alpha$ ;  $E_\alpha^{(s)}$ ,  $s = 1, \dots, m_\alpha$ , are copies of the space  $E_\alpha$  of the representation  $\alpha$ ,  $\underline{m} := \{m_\alpha\}$ ). Let  $f_{\alpha,\ell}$  with  $\alpha \in r(H)$ ,  $\ell = 1, \dots, \ell_\alpha$ , be  $H$ -equivariant germs of holomorphic functions (maps)  $(\mathbb{C}^N, 0) \rightarrow (E_\alpha, 0)$  (i.e.  $f_{\alpha,\ell}(gx) = \alpha(g)f_{\alpha,\ell}(x)$  for  $x \in \mathbb{C}^N$ ,  $g \in H$ ) such that  $(V, 0) = \{f_{\alpha,\ell} = 0 \mid \alpha \in r(H), \ell = 1, \dots, \ell_\alpha\}$  is an isolated complete intersection singularity in  $(\mathbb{C}^N, 0)$  of codimension  $\sum_{\alpha} \ell_\alpha \dim E_\alpha$ .

The differentials  $df_{\alpha,\ell}$  of the functions  $f_{\alpha,\ell}$  (with values in  $E_\alpha$ ) at a point  $p \in V \cap m_1 E_1$  is an  $H$ -equivariant linear map from  $T\mathbb{C}^N (\cong \mathbb{C}^N)$  to  $E_\alpha$ . According to Schur's lemma  $df_{\alpha,\ell}$  vanishes on  $\bigoplus_{\beta \neq \alpha} m_\beta E_\beta$  and is the multiplication by a complex number (depending on the point  $p$ ) on each copy  $E_\alpha^{(s)}$ ,  $s = 1, \dots, m_\alpha$ . Let us denote this number by  $\partial_\alpha^s f_{\alpha,\ell}(p)$ . Thus one has

$$df_{\alpha,\ell}|_p([u_\beta^s]) = \sum_s \partial_\alpha^s f_{\alpha,\ell}(p) u_\alpha^s.$$

For a collection  $\underline{k} = \{k^{\alpha,i}\}$  with  $\sum_{\alpha,i} k^{\alpha,i} = m_1 - \ell_1$ ,  $k_i^\alpha \leq n_\alpha := m_\alpha - \ell_\alpha$

for all  $(\alpha, i)$ , let  $\omega_j^{\alpha,i}$ ,  $j = 1, \dots, n_\alpha - k_{\alpha,i} + 1$ , be  $H$ -equivariant 1-forms on a neighbourhood of the origin in  $(\mathbb{C}^N, 0)$  with values in the spaces  $E_\alpha$ . For  $p \in E_1^{m_1} \cap V$ , let  $\omega_{j|p}^{\alpha,i}([u_s^\alpha]) = \sum_s \psi_{j,s}^{\alpha,i}(p) u_s^\alpha$ . Let us define a map  $\Psi : E_1^{m_1} \cap V \rightarrow M_{\underline{m}, \underline{k}}$  by

$$\Psi(p) = \left\{ \begin{pmatrix} \partial_\alpha^1 f_{\alpha,1}(p) & \cdots & \partial_\alpha^{\ell_\alpha} f_{\alpha,\ell_\alpha}(p) & \psi_{1,1}^{\alpha,i}(p) & \cdots & \psi_{n_\alpha - k_{\alpha,i} + 1,1}^{\alpha,i}(p) \\ \vdots & \cdots & \vdots & \vdots & \cdots & \vdots \\ \partial_\alpha^{m_\alpha} f_{\alpha,1}(p) & \cdots & \partial_\alpha^{m_\alpha} f_{\alpha,\ell_\alpha}(p) & \psi_{1,m_\alpha}^{\alpha,i}(p) & \cdots & \psi_{n_\alpha - k_{\alpha,i} + 1, m_\alpha}^{\alpha,i}(p) \end{pmatrix} \right\},$$

where the right hand side of the equation is the collection of  $m_\alpha \times (m_\alpha - k_{\alpha,i} + 1)$  matrices  $A_{\alpha,i}$  whose first  $\ell_\alpha$  columns consist of the components of the differentials  $df_{\alpha,\ell}|_p$  of the functions  $f_{\alpha,\ell}|_p$  and the last  $n_\alpha - k_{\alpha,i} + 1$  columns consist of the components  $\psi_{j,s}^{\alpha,i}(p)$  of the 1-forms  $\omega_{j|p}^{\alpha,i}$ . Assume that the collection of the forms has no singular points on  $V$  outside of the origin (in a neighbourhood of it). This means that  $\Psi^{-1}(D_{\underline{m}, \underline{k}}) = \{0\}$ .

**Definition:** The *equivariant GSV-index*  $\text{ind}_{V,0}^{H,\underline{n},\underline{k}}\{\omega_j^{\alpha,i}\}$  of the collection  $\{\omega_j^{\alpha,i}\}$  on the ICIS  $(V, 0)$  is the degree of the map  $\Psi|_{E_1^{m_1} \cap V \cap S_\varepsilon^{2N-1}} : E_1^{m_1} \cap V \cap S_\varepsilon^{2N-1} \rightarrow W_{\underline{n},\underline{k}}$  (or, what is the same, the intersection number of the image  $\Psi(E_1^{m_1} \cap V)$  with  $D_{\underline{n},\underline{k}}$  at the origin).

Being an intersection number, the equivariant GSV-index satisfies the law of conservation of number. This means that if  $\tilde{V} = V_\lambda$  is an ( $H$ -invariant) deformation of the ICIS  $V$  and a collection  $\{\tilde{\omega}_j^{\alpha,i}\} = \{\omega_{j;\lambda}^{\alpha,i}\}$  is a deformation of the collection  $\{\omega_j^{\alpha,i}\}$  with isolated singular points on  $V_\lambda$ , then, for  $\lambda$  small enough, one has

$$\text{ind}_{V,0}^{H,\underline{n},\underline{k}}\{\omega_j^{\alpha,i}\} = \sum_Q \text{ind}_{\tilde{V},Q}^{H,\underline{n},\underline{k}}\{\tilde{\omega}_j^{\alpha,i}\},$$

where the sum on the right hand side runs over all singular points  $Q$  of the collection  $\{\tilde{\omega}_j^{\alpha,i}\}$  on  $\tilde{V}$  in a neighbourhood of the origin (including the singular points of  $\tilde{V}$  itself).

This implies that, for a compact  $H$ -variety  $V$  with only isolated ( $H$ -invariant) complete intersection singularities and for a collection of  $H$ -equivariant forms  $\{\omega_j^{\alpha,i}\}$  on it with only isolated singular points, the sum

$$\sum_Q \text{ind}_{V,Q}^{H,\underline{n},\underline{k}}\{\omega_j^{\alpha,i}\}$$

of the equivariant GSV-indices of the collection  $\{\omega_j^{\alpha,i}\}$  over all its singular points  $Q$  does not depend on the collection and is an invariant of  $V$ . It can be considered as an analogue of the corresponding Chern number. Moreover, this Chern number is the same for varieties with only isolated ( $H$ -invariant) complete intersection singularities from a, say, one-parameter family of them. In particular, if this family includes (as the generic member) a smooth variety, the defined Chern number coincides with the usual one.

Assume that all the forms  $\omega_j^{\alpha,i}$ ,  $\alpha \in r(H)$ ,  $i = 1, \dots, s_\alpha$ ,  $j = 1, \dots, n_\alpha - k_{\alpha,i} + 1$ , are complex analytic (in a neighbourhood of the origin in  $\mathbb{C}^N$ ). Let  $I_{V,\{\omega_j^{\alpha,i}\}}$  be the ideal of the ring  $\mathcal{O}_{\mathbb{C}^{m_1},0}$  generated by the germs  $f_{1,1}|_{\mathbb{C}^{m_1}}, \dots, f_{1,\ell_1}|_{\mathbb{C}^{m_1}}$  and by the maximal (i.e. of size  $(m_\alpha - k_{\alpha,i} + 1) \times (m_\alpha - k_{\alpha,i} + 1)$ ) minors of the matrices  $A_{\alpha,i}$  for all  $(\alpha, i)$ .

**Theorem 1** *One has*

$$\text{ind}_{V,0}^{H,\underline{n},\underline{k}}\{\omega_j^{\alpha,i}\} = \dim_{\mathbb{C}} \mathcal{O}_{\mathbb{C}^{m_1},0} / I_{V,\{\omega_j^{\alpha,i}\}}.$$

The proof is essentially the same as in [3].

### 3 Equivariant Chern obstructions and characteristic numbers

Let the group  $H$  act (by a representation) on the affine space  $\mathbb{C}^N$  and let  $(X, 0) \subset (\mathbb{C}^N, 0)$  be an  $H$ -invariant (reduced) germ of a complex analytic variety of (pure) dimension  $n$ . Let  $\pi : \widehat{X} \rightarrow X$  be the Nash transformation of the variety  $X \subset B_\varepsilon$  ( $B_\varepsilon = B_\varepsilon^{2N}$  is the ball of a sufficiently small radius  $\varepsilon$  around the origin in  $\mathbb{C}^N$ ) defined as follows. Let  $X_{\text{reg}}$  be the set of smooth points of  $X$  and let  $G(n, N)$  be the Grassmann manifold of  $n$ -dimensional vector subspaces of  $\mathbb{C}^N$ . The action of the group  $H$  on  $\mathbb{C}^N$  defines an action on  $G(n, N)$  and on the space of the tautological bundle over it as well. There is a natural map  $\sigma : X_{\text{reg}} \rightarrow B_\varepsilon \times G(n, N)$  which sends a point  $x \in X_{\text{reg}}$  to  $(x, T_x X_{\text{reg}})$ . The Nash transform  $\widehat{X}$  of the variety  $X$  is the closure of the image  $\text{Im } \sigma$  of the map  $\sigma$  in  $B_\varepsilon \times G(n, N)$ ,  $\pi$  is the natural projection. The Nash bundle  $\widehat{T}$  over  $\widehat{X}$  is a vector bundle of rank  $n$  which is the pullback of the tautological bundle on the Grassmann manifold  $G(n, N)$ . There is a natural lifting of the Nash transformation to a bundle map from the Nash bundle  $\widehat{T}$  to the restriction of the tangent bundle  $T\mathbb{C}^N$  of  $\mathbb{C}^N$  to  $X$ . This is an isomorphism of  $\widehat{T}$  and  $TX_{\text{reg}} \subset T\mathbb{C}^N$  over the non-singular part  $X_{\text{reg}}$  of  $X$ .

For a collection of integers  $\underline{n} = \{n_\alpha\}$ , let  $X^{\underline{n}}$  be the closure in  $X$  of the set of points  $x \in X_{\text{reg}} \cap X^H$  such that the representation of the group  $H$  on the tangent space  $T_x X_{\text{reg}}$  is  $\bigoplus_\alpha n_\alpha \alpha$  and let  $\widehat{X}^{\underline{n}}$  be the closure of the same set in  $\widehat{X}$ . The restriction of the Nash bundle  $\widehat{T}$  to  $\widehat{X}^{\underline{n}}$  is the direct sum of the subbundles  $\widehat{T}_\alpha$  ( $\alpha \in r(H)$ ) subject to the splitting of the representation of  $H$  on  $\widehat{T}$  into the summands corresponding to different irreducible representations of  $H$ . Let the (complex) vector bundle  $\widehat{\nu}_\alpha^*$  of rank  $n_\alpha$  over  $\widehat{X}^{\underline{n}}$  be defined by

$$\widehat{\nu}_\alpha^* := \text{Hom}_H(\widehat{T}, E_\alpha) = \text{Hom}_H(\widehat{T}_\alpha, E_\alpha).$$

Let  $\underline{k} = \{k_{\alpha,i}\}$ ,  $\alpha \in r(H)$ ,  $i = 1, \dots, n_\alpha - k_{\alpha,i} + 1$ , be a collection of positive integers with  $\sum_{\alpha,i} k_{\alpha,i} = n_1$  and let  $\{\omega_j^{\alpha,i}\}$ ,  $j = 1, \dots, n_\alpha - k_{\alpha,i} + 1$ , be a collection of  $H$ -equivariant 1-forms on  $(\mathbb{C}^N, 0)$  with values in the spaces  $E_\alpha$  of the representations  $(\omega_j^{\alpha,i}(gu) = \alpha(g)\omega_j^{\alpha,i}(u)$  for  $g \in H$ ).

**Remark 1** One can see that all the constructions below are determined by the restrictions of the forms  $\omega_j^{\alpha,i}$  to the regular part of the variety  $X$ .

Let  $\varepsilon > 0$  be small enough so that there is a representative  $X$  of the germ  $(X, 0)$  and representatives  $\omega_j^{\alpha,i}$  of the germs of 1-forms inside the ball  $B_\varepsilon \subset \mathbb{C}^N$ .



**Definition:** A point  $P \in X^n$  is called a *special* point of the collection  $\{\omega_j^{\alpha,i}\}$  of 1-forms on the variety  $X$  if there exists a sequence  $\{P_m\}$  of points from  $X^n \cap X_{\text{reg}}$  converging to  $P$  such that the sequence  $T_{P_m} X_{\text{reg}}$  of the tangent spaces at the points  $P_m$  has an ( $H$ -invariant) limit  $L$  as  $m \rightarrow \infty$  (in the Grassmann manifold of  $n$ -dimensional vector subspaces of  $\mathbb{C}^N$ ) and the restrictions of the 1-forms  $\omega_1^{\alpha,i}, \dots, \omega_{n_\alpha - k_{\alpha,i} + 1}^{\alpha,i}$  to the subspace  $L \subset T_P \mathbb{C}^N$  are linearly dependent for each pair  $(\alpha, i)$ .

**Definition:** The collection  $\{\omega_j^{\alpha,i}\}$  of 1-forms has an *isolated special point* on the germ  $(X, 0)$  if it has no special points on  $X$  in a punctured neighbourhood of the origin.

**Remark 2** On a smooth variety the notions “special point” and “singular point” coincide. (On a singular variety these notions are different: see [4, Remark 1.2].) A singular point of a collection of 1-forms  $\{\omega_j^{\alpha,i}\}$  on a variety can be non-degenerate only if it is a smooth point of the variety and therefore it is a special point as well.

Let

$$\mathcal{L}^{n,k} = \prod_{\alpha,i} \prod_{j=1}^{n_\alpha - k_{\alpha,i} + 1} \text{Hom}_H(\mathbb{C}^N, E_\alpha)$$

be the space of collections of  $E_\alpha$ -valued linear functions on  $\mathbb{C}^N$  (i.e. of  $H$ -equivariant 1-forms with constant coefficients).

**Proposition 2** *There exists an open and dense subset  $U \subset \mathcal{L}^{n,k}$  such that each collection  $\{\lambda_j^{\alpha,i}\} \in U$  has only isolated special points on  $X^n$  and, moreover, all these points belong to the smooth part  $X^n \cap X_{\text{reg}}$  of the variety  $X^n$  and are non-degenerate.*

*Proof.* Let  $Y \subset X^n \times \mathcal{L}^{n,k}$  be the closure of the set of pairs  $(x, \{\lambda_j^{\alpha,i}\})$  where  $x \in X^n \cap X_{\text{reg}}$  and the restrictions of the linear functions  $\lambda_1^{\alpha,i}, \dots, \lambda_{n_\alpha - k_{\alpha,i} + 1}^{\alpha,i}$  to the tangent space  $T_x X_{\text{reg}}$  are linearly dependent for each pair  $(\alpha, i)$ . Let  $\text{pr}_2 : Y \rightarrow \mathcal{L}^{n,k}$  be the projection to the second factor. One has  $\text{codim } Y = \sum_{\alpha,i} k_{\alpha,i} = n_1$  and therefore  $\dim Y = \dim \mathcal{L}^{n,k}$ .

Moreover,  $Y \setminus ((X^n \cap X_{\text{reg}}) \times \mathcal{L}^{n,k})$  is a proper subvariety of  $Y$  and therefore its dimension is strictly smaller than  $\dim \mathcal{L}^{n,k}$ . A generic point  $\Lambda = \{\lambda_j^{\alpha,i}\}$  of the space  $\mathcal{L}^{n,k}$  is a regular value of the map  $\text{pr}_2$  which means that it has only finitely many preimages, all of them belong to  $(X^n \cap X_{\text{reg}}) \times \mathcal{L}^{n,k}$  and the map  $\text{pr}_2$  is non-degenerate at them. Therefore  $\Lambda \times (X^n \cap X_{\text{reg}})$  intersects  $Y$  transversally. This implies the statement.  $\square$

**Corollary 1** *Let  $\{\omega_j^{\alpha,i}\}$  be a collection of 1-forms on  $X$  with an isolated special point at the origin. Then there exists a deformation  $\{\tilde{\omega}_j^{\alpha,i}\}$  of the collection  $\{\omega_j^{\alpha,i}\}$  whose special points lie in  $X^n \cap X_{\text{reg}}$  and are non-degenerate. Moreover, as such a deformation one can use  $\{\omega_j^{\alpha,i} + \lambda_j^{\alpha,i}\}$  with a generic collection  $\{\lambda_j^{\alpha,i}\} \in \mathcal{L}^{n,k}$  small enough.*

Let

$$\widehat{\mathbb{T}}^{n,k} = \bigoplus_{\alpha,i} \bigoplus_{j=1}^{n_\alpha - k_{\alpha,i} + 1} \widehat{\nu}_{\alpha,i,j}^*,$$

where  $\widehat{\nu}_{\alpha,i,j}^*$  are copies of the vector bundle  $\widehat{\nu}_\alpha^*$  numbered by  $i$  and  $j$ . Let  $\widehat{\mathbb{D}}^{n,k} \subset \widehat{\mathbb{T}}^{n,k}$  be the set of pairs  $(x, \{\eta_j^{\alpha,i}\})$ ,  $x \in \widehat{X}^n$ ,  $\eta_j^{\alpha,i} \in \widehat{\nu}_{\alpha,i,j}^*$ , such that  $\eta_1^{\alpha,i}, \dots, \eta_{n_\alpha - k_{\alpha,i} + 1}^{\alpha,i}$  are linearly dependent for each pair  $(\alpha, i)$ .

The collection  $\{\omega_j^{\alpha,i}\}$  defines a section  $\widehat{\omega}$  of the bundle  $\widehat{\mathbb{T}}^{n,k}$ . The image of this section does not intersect  $\widehat{\mathbb{D}}^{n,k}$  outside of the preimage  $\pi^{-1}(0) \subset \widehat{X}^n$  of the origin. The map  $\widehat{\mathbb{T}}^{n,k} \setminus \widehat{\mathbb{D}}^{n,k} \rightarrow \widehat{X}^n$  is a fibre bundle. The fibre  $W_x = \widehat{\mathbb{T}}_x^{n,k} \setminus \widehat{\mathbb{D}}_x^{n,k}$  of it is  $(2n_1 - 2)$ -connected, its homology group  $H_{2n_1-1}(W_x; \mathbb{Z})$  is isomorphic to  $\mathbb{Z}$  and has a natural generator. (This follows from the fact that  $W_x$  is homeomorphic to  $M_{n,k} \setminus D_{n,k}$  from Section 1.) The latter fact implies that the fibre bundle  $\widehat{\mathbb{T}}^{n,k} \setminus \widehat{\mathbb{D}}^{n,k} \rightarrow \widehat{X}^n$  is homotopically simple in dimension  $2n_1 - 1$ , i.e. the fundamental group  $\pi_1(\widehat{X}^n)$  of the base acts trivially on the homotopy group  $\pi_{2n_1-1}(W_x)$  of the fibre, the last one being isomorphic to the homology group  $H_{2n_1-1}(W_x; \mathbb{Z})$ : see, e.g., [9].

**Definition:** The *local Chern obstruction (index)*  $\text{Ch}_{X,0}^{H,n,k} \{\omega_j^{\alpha,i}\}$  of the collections of germs of 1-forms  $\{\omega_j^{\alpha,i}\}$  on  $(X, 0)$  at the origin is the (primary, and in fact the only) obstruction to extend the section  $\widehat{\omega}$  of the fibre bundle  $\widehat{\mathbb{T}}^{n,k} \setminus \widehat{\mathbb{D}}^{n,k} \rightarrow \widehat{X}^n$  from the preimage of a neighbourhood of the sphere  $S_\varepsilon = \partial B_\varepsilon$  to  $\widehat{X}^n$ , more precisely its value (as an element of  $H^{2n_1}(\pi^{-1}(X^n \cap B_\varepsilon), \pi^{-1}(X^n \cap S_\varepsilon); \mathbb{Z})$ ) on the fundamental class of the pair  $(\pi^{-1}(X^n \cap B_\varepsilon), \pi^{-1}(X^n \cap S_\varepsilon))$ .

The definition of the local Chern obstruction  $\text{Ch}_{X,0}^{H,n,k} \{\omega_j^{\alpha,i}\}$  can be reformulated in the following way. The collection of 1-forms  $\{\omega_j^{\alpha,i}\}$  defines also a section  $\tilde{\omega}$  of the trivial bundle  $X^n \times \mathcal{L}^{n,k} \rightarrow X^n$ , namely,  $\tilde{\omega} : X^n \rightarrow X^n \times \mathcal{L}^{n,k}$ . Let  $\mathcal{D}^{n,k} \subset X^n \times \mathcal{L}^{n,k}$  be the closure of the set of pairs  $(x, \{\lambda_j^{\alpha,i}\})$  such that  $x \in X^n \cap X_{\text{reg}}$  and the restrictions of the linear functions  $\lambda_1^{\alpha,i}, \dots, \lambda_{n_\alpha - k_{\alpha,i} + 1}^{\alpha,i}$  to  $T_x X_{\text{reg}} \subset \mathbb{C}^N$  are linearly dependent for each pair  $(\alpha, i)$ . For  $x \in X^n \cap S_\varepsilon$  ( $\varepsilon$  small enough),  $\tilde{\omega} \notin \mathcal{D}^{n,k}$  and the local Chern obstruction is the value on the fundamental class of the pair  $(X^n \cap B_\varepsilon, X^n \cap S_\varepsilon)$  of the first obstruction to extend the map  $\tilde{\omega} : X^n \cap S_\varepsilon \rightarrow (X^n \times \mathcal{L}^{n,k}) \setminus \mathcal{D}^{n,k}$  to a map  $X^n \cap B_\varepsilon \rightarrow (X^n \times \mathcal{L}^{n,k}) \setminus \mathcal{D}^{n,k}$ . (This follows, e.g. from the fact

that, due to Corollary 1, the collection  $\{\omega_j^{\alpha,i}\}$  can be deformed in such a way that all the special points of the deformed collection lie in the regular part of  $X^{\underline{n}}$  and thus the corresponding obstructions on  $X^{\underline{n}}$  and on  $\widehat{X}^{\underline{n}}$  coincide.) The map  $\tilde{\omega}$  is also defined on  $\mathbb{C}^N$  (a section of the trivial bundle  $\mathbb{C}^N \times \mathcal{L}^{\underline{n},k} \rightarrow \mathbb{C}^N$ ):  $\tilde{\omega} : \mathbb{C}^N \rightarrow \mathbb{C}^N \times \mathcal{L}^{\underline{n},k}$ . For  $x \in S_\varepsilon$  ( $\varepsilon$  small enough),  $\tilde{\omega} \notin \mathcal{D}^{\underline{n},k} \subset X^{\underline{n}} \times \mathcal{L}^{\underline{n},k} \subset \mathbb{C}^N \times \mathcal{L}^{\underline{n},k}$  and the local Chern obstruction is the value on the fundamental class of the pair  $(B_\varepsilon, S_\varepsilon)$  of the first obstruction to extend the map  $\tilde{\omega} : S_\varepsilon \rightarrow (\mathbb{C}^N \times \mathcal{L}^{\underline{n},k}) \setminus \mathcal{D}^{\underline{n},k}$  to a map  $B_\varepsilon \rightarrow (\mathbb{C}^N \times \mathcal{L}^{\underline{n},k}) \setminus \mathcal{D}^{\underline{n},k}$ . This means that it is equal to the intersection number  $(\tilde{\omega}(\mathbb{C}^N) \circ \mathcal{D}^{\underline{n},k})_0$  at the origin in  $\mathbb{C}^N \times \mathcal{L}^{\underline{n},k}$ .

Being a (primary) obstruction, the local Chern obstruction satisfies the law of conservation of number, i.e. if a collection of 1-forms  $\{\tilde{\omega}_j^{\alpha,i}\}$  is a deformation of the collection  $\{\omega_j^{\alpha,i}\}$  with only isolated special points on  $X$ , then

$$\text{Ch}_{X,0}^{H,\underline{n},k} \{\omega_j^{\alpha,i}\} = \sum_Q \text{Ch}_{X,Q}^{H,\underline{n},k} \{\tilde{\omega}_j^{\alpha,i}\},$$

where the sum on the right hand side is over all special points  $Q$  of the collection  $\{\tilde{\omega}_j^{\alpha,i}\}$  on  $X$  in a neighbourhood of the origin.

Along with Corollary 1 this implies the following statements.

**Proposition 3** *The local Chern obstruction  $\text{Ch}_{X,0}^{H,\underline{n},k} \{\omega_j^{\alpha,i}\}$  of a collection  $\{\omega_j^{\alpha,i}\}$  of germs of 1-forms is equal to the algebraic (i.e. counted with signs) number of special points on  $X$  of a generic deformation of the collection.*

If all the 1-forms  $\omega_j^{\alpha,i}$  and their generic deformations are holomorphic, the local Chern obstructions of the deformed collection at its special points are equal to 1 and therefore the local Chern obstruction  $\text{Ch}_{X,0}^{H,\underline{n},k} \{\omega_j^{\alpha,i}\}$  is equal to the number of special points of the deformation.

**Proposition 4** *If a collection  $\{\omega_j^{\alpha,i}\}$  ( $\alpha \in r(H)$ ,  $i = 1, \dots, s_\alpha$ ,  $j = 1, \dots, n_\alpha - k_{\alpha,i} + 1$ ) of 1-forms on a compact (say, projective)  $G$ -variety  $X$  has only isolated special points on  $X^H$ , then the sum of the local Chern obstructions of the collection  $\{\omega_j^{\alpha,i}\}$  at these points does not depend on the collection and therefore is an invariant of the variety.*

One can consider this sum as the corresponding Chern number of the  $G$ -variety  $X$ .

Let  $(X, 0)$  be an isolated  $H$ -invariant complete intersection singularity (see Section 2). As it was described there, a collection of germs of 1-forms  $\{\omega_j^{\alpha,i}\}$  on  $(X, 0)$  with an isolated special point at the origin has an index  $\text{ind}_{X,0}^{H,\underline{n},k} \{\omega_j^{\alpha,i}\}$  which is an analogue of the GSV-index of a 1-form. The fact that both the

Chern obstruction and the index satisfy the law of conservation of number and they coincide on a smooth manifold yields the following statement.

**Proposition 5** *For a collection  $\{\omega_j^{\alpha,i}\}$  of germs of 1-forms on an isolated  $H$ -invariant complete intersection singularity  $(X, 0)$  the difference*

$$\mathrm{ind}_{X,0}^{H,\underline{n},\underline{k}} \{\omega_j^{\alpha,i}\} - \mathrm{Ch}_{X,0}^{H,\underline{n},\underline{k}} \{\omega_j^{\alpha,i}\}$$

*does not depend on the collection and therefore is an invariant of the germ of the variety.*

Since, by Proposition 2,  $\mathrm{Ch}_{X,0}^{H,\underline{n},\underline{k}} \{\omega_j^{\alpha,i}\} = 0$  for a generic collection  $\{\omega_j^{\alpha,i}\}$  of linear functions on  $\mathbb{C}^N$ , one has the following statement.

**Corollary 2** *One has*

$$\mathrm{Ch}_{X,0}^{H,\underline{n},\underline{k}} \{\omega_j^{\alpha,i}\} = \mathrm{ind}_{X,0}^{H,\underline{n},\underline{k}} \{\omega_j^{\alpha,i}\} - \mathrm{ind}_{X,0}^{H,\underline{n},\underline{k}} \{\lambda_j^{\alpha,i}\}$$

*for a generic collection  $\{\lambda_j^{\alpha,i}\}$  of  $H$ -equivariant linear functions on  $\mathbb{C}^N$ .*

**Remark 3** Instead of working with  $E_\alpha$ -valued 1-forms on a compact variety  $X$ , one can also consider 1-forms with values "in local systems of coefficients", i.e. in  $(H-)$ vector bundles of the form  $E_\alpha \otimes L_\alpha$  where  $L_\alpha$  is a (usual) line bundle over  $X$ . The local indices (both GSV- and Chern ones) of a collection of 1-forms are defined in the same way and the sums of these indices give invariants of a compact  $G$ -variety  $X$  with the bundles  $L_\alpha$  which can be regarded as analogues of the Chern numbers  $\langle \prod_{\alpha,i} c_{k_{\alpha,i}}(\nu_\alpha^* \otimes L_\alpha), [X] \rangle$  (see the Introduction).

**Remark 4** The definitions and constructions of this section work for  $G$  being a compact Lie group as well.

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